General Relativity

We describe a point in spacetime by its position vector x^μ where $\mu = 0,1,2,3$:

$$x^\mu = (\underset{\uparrow\, x^0}{ct}, x^i) \quad \text{where} \quad i = 1,2,3.$$

The spacetime interval ds^2 between two points can be written as

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$= g_{\mu\nu}\, dx^\mu\, dx^\nu$$ Einstein summation notation

where $g_{\mu\nu}$ is the metric tensor. The metric tensor depends on the geometry of spacetime and the coordinates used.

The metric has 16 components, but they are not all independent. The metric is symmetric, i.e., $g_{\mu\nu} = g_{\nu\mu}$, so there are 10 independent degrees of freedom.

ex. In flat spacetime, the spacetime interval is

$$ds^2 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{(Cartesian coordinates)}$$

$$dx^\mu = \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix} \qquad ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \quad \text{where} \quad \underbrace{\eta_{\mu\nu}}_{\text{Minkowski metric}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ex. Outside of a spherically-symmetric body with mass M (non-rotating),

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \quad \text{where} \quad \underbrace{g_{\mu\nu}}_{\text{Schwarzschild metric}} = \begin{pmatrix} -\left(1-\frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Einstein Field Equations

The Einstein Field Equations relate the curvature of spacetime to the distribution of mass/energy/momentum:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

stress-energy tensor

Einstein curvature tensor

$G_{\mu\nu}$ is the Einstein curvature tensor and is built from the metric and its derivatives (1st and 2nd).

Compare this to Maxwell's equations:

$$\nabla \cdot \vec{E} = 4\pi\rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

LHS: fields

RHS: sources

Linearized Gravity

Suppose spacetime is nearly but not quite flat. Then the metric can be written as

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Minkowski}} + h_{\mu\nu}$$

small perturbation

Define $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$.

trace-reversed metric perturbation

Then the linearized Einstein Field Equations are

$$-\eta^{\alpha\beta} \frac{\partial^2 \bar{h}_{\mu\nu}}{\partial x^\alpha \, \partial x^\beta} + \mathcal{O}(h^2) = \frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\Rightarrow \; -\Box \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} \qquad \text{where} \quad \Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

Note that we have enforced the gauge condition $\frac{\partial}{\partial x^\mu} \bar{h}^{\mu\alpha} = 0$ (Lorenz gauge).

This is analogous to the Lorenz gauge in electromagnetism $\frac{\partial}{\partial x^\mu} A^\mu = 0$

Gravitational Waves

In vacuum, the linearized Einstein Field Equations are

$$\Box \bar{h}_{\mu\nu} = 0 \qquad\qquad \frac{\partial \bar{h}^{\alpha\mu}}{\partial x^\alpha} = 0 \quad \text{(Lorenz gauge)}$$

This equation admits plane-wave solutions:

$$\bar{h}_{\mu\nu}(x^\alpha) = \bar{h}_{\mu\nu}(t - z/c)$$

wave traveling in the z-direction

The Lorenz gauge condition imposes 4 constraints:

$$\frac{\partial \bar{h}^{\alpha\mu}}{\partial x^\alpha} = 0 \qquad \text{for } \mu = 0, 1, 2, 3$$

so the number of independent components of $\bar{h}_{\mu\nu}$ is $10 - 4 = 6$.

But not all of these correspond to physical degrees of freedom.

Recall that curvature is calculated from 2nd derivatives of the metric. For our plane wave solution, only $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ are nonzero. We can use this to show that there are only 2 physical degrees of freedom:

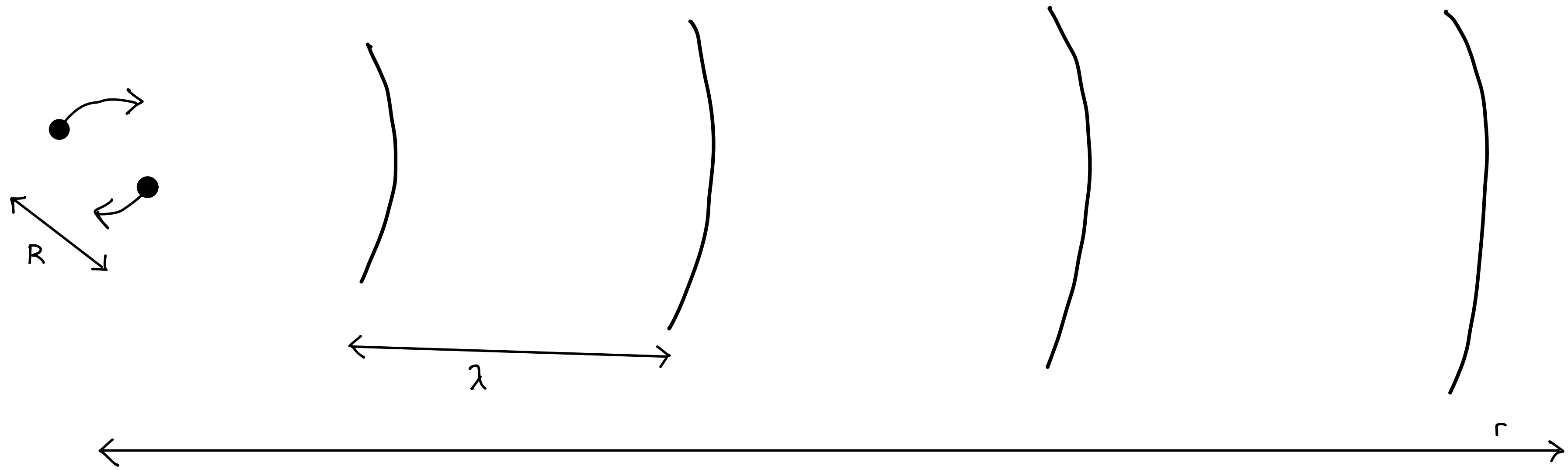
$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + h_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

transverse-traceless (TT) gauge

Gravitational waves carry away energy. If we look at the Einstein field equations at $\mathcal{O}(h^2)$, we find an effective stress-energy tensor from the GWs:

$$T_{\mu\nu}^{GW} = \frac{c^4}{8\pi G} \left\langle \frac{\partial h^{ij}_{TT}}{\partial x^\mu} \frac{\partial h_{ij}^{TT}}{\partial x^\nu} \right\rangle$$

Astrophysical Sources of GWs



Relativistic binaries produce gravitational waves:

$$\bar{h}^{ij}(t,\vec{x}) = \frac{4G}{c^4}\int d^3x' \, \frac{\tau^{ij}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|}$$

$R \ll \lambda \ll r$

$$\approx \frac{4G}{c^4 r}\int d^3x' \, \tau^{ij}\left(t-\frac{r}{c}, \vec{x}'\right)$$

distribution of mass

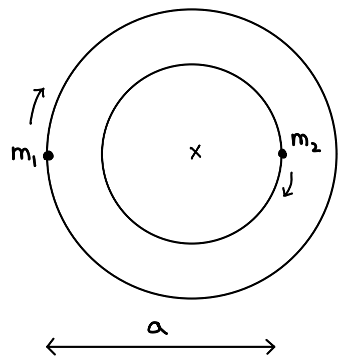
$$\approx \frac{2G}{c^4 r}\ddot{I}^{ij}\left(t-\frac{r}{c}\right) \qquad \text{where } I^{ij} := \int d^3x \; \tau^{00}(t,\vec{x})\, x^i x^j$$

quadrupole tensor

The energy carried away is $\mathcal{L} = -\frac{dE}{dt} \sim \frac{G}{c^5}\dddot{I}^2$

$$\sim \frac{G}{c^5}\left(\frac{m_1 m_2}{M} a^2 \omega^3\right)^2$$

$$\sim \frac{G}{c^5}\left(\frac{m_1 m_2}{M}\right)^2 a^4 \omega^6$$



$$\vec{r}_1 = \frac{m_2}{M} a\left(-\cos \omega t\ \hat{x} + \sin \omega t\ \hat{y}\right)$$

$$\vec{r}_2 = \frac{m_1}{M} a\left(\cos \omega t\ \hat{x} - \sin \omega t\ \hat{y}\right)$$

This causes the orbit to decay and the orbital frequency to increase:

$$E = -\frac{G m_1 m_2}{2a} \Rightarrow \frac{da}{dt} = \frac{2a^2}{G m_1 m_2}\frac{dE}{dt}$$

$$\frac{da}{dt} \sim -\frac{1}{c^5}\frac{m_1 m_2}{M^2} a^6 \omega^6$$

Kepler's 3rd Law $\omega^2 a^3 = GM$

$$\frac{d\omega}{dt} \sim \frac{G^{5/3}}{c^5}\mathcal{M}_c^{5/3}\omega^{11/3} \quad \text{where} \quad \mathcal{M}_c \equiv \frac{(m_1 m_2)^{3/5}}{M^{1/5}}$$

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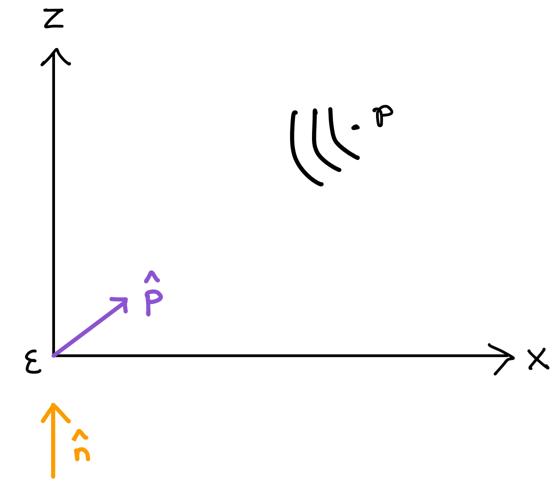
The GW frequency is related to the orbital frequency by a factor of 2:

$$f_{GW} = 2 f_{orb} = \frac{\omega}{\pi}.$$

Detecting Gravitational Waves



You are observing signal from an emitter $\mathcal{P}$ at the Earth $\mathcal{E}$.

Gravitational waves are propagating in the direction $\hat{n} = \hat{z}$.

Consider just the + polarization:

$$h_{11} = -h_{22} = h_+(t - z/c).$$

The signal has the frequency 4-vector k^α, which is a null vector:

$$(\eta_{\mu\nu} + h_{\mu\nu})k^\mu k^\nu = 0$$

$$\Rightarrow \quad k^\alpha = \nu c \begin{pmatrix} 1 \\ -\left(1 - \frac{1}{2}h_+\right)\sin\theta \\ 0 \\ \cos\theta \end{pmatrix} \qquad \text{where} \quad \cos\theta = \hat{n}\cdot\hat{p}$$

The radio signal follows a geodesic:

$$\frac{dk^\alpha}{d\lambda} = -\Gamma^\alpha_{\mu\nu} k^\mu k^\nu$$

Christoffel symbols

The redshift from gravitational waves is

$$\frac{d\nu}{d\lambda} = \frac{1}{c}\frac{dk^0}{d\lambda} = -\frac{1}{c}\Gamma^0_{\mu\nu} k^\mu k^\nu$$

Christoffel symbols are built from derivatives of the metric

$$= -\frac{1}{2}\frac{\partial h_+}{\partial t}\nu^2 \sin^2\theta$$

use chain rule to change from $\frac{\partial h_+}{\partial t}$ to $\frac{dh_+}{d\lambda}$

$$= -\frac{1}{2}\left[\frac{1}{\nu(1+\cos\theta)}\frac{dh_+}{d\lambda}\right]\nu^2\sin^2\theta$$

$$= -\frac{1}{2}\frac{\sin^2\theta}{1+\cos\theta}\frac{dh_+}{d\lambda}\nu$$

$$\Rightarrow \frac{1}{\nu}\frac{d\nu}{d\lambda} = -\frac{1}{2}(1-\cos\theta)\frac{dh_+}{d\lambda}$$

$$\Rightarrow \frac{d\ln\nu}{d\lambda} = -\frac{d}{d\lambda}\left[\frac{1}{2}(1-\cos\theta)h_+\right]$$

Integrating along the line of sight gives

$$-\ln \nu \Big|_p^{\varepsilon} = \int_p^{\varepsilon} d\lambda \, \frac{d}{d\lambda}\left[\frac{1}{2}(1-\cos\theta)\, h_+\right] = \frac{1}{2}(1-\cos\theta)\left[h_+(x^\mu_\varepsilon) - h_+(x^\mu_p)\right]$$

$$= \ln \frac{\nu_p}{\nu_\varepsilon} = \ln\left(1 + \frac{\nu_p - \nu_\varepsilon}{\nu_\varepsilon}\right) \approx \frac{\nu_p - \nu_\varepsilon}{\nu_\varepsilon}$$

$$\Rightarrow \frac{\nu_p - \nu_\varepsilon}{\nu_\varepsilon} = \frac{1}{2}(1-\cos\theta)\left[h_+(x^\mu_\varepsilon) - h_+(x^\mu_p)\right]$$

Generalizing to + and × polarizations, and GWs propagating in other directions,

$$\frac{\nu_p - \nu_\varepsilon}{\nu_\varepsilon} = \frac{1}{2}\frac{\hat{p}^i \hat{p}^j}{1+\hat{p}\cdot\hat{\Omega}}\Big\{ e^+_{ij}(\hat{\Omega})\left[h_+(x^\mu_\varepsilon,\hat{\Omega}) - h_+(x^\mu_p,\hat{\Omega})\right]$$

$$+ e^\times_{ij}(\hat{\Omega})\left[h_\times(x^\mu_\varepsilon,\hat{\Omega}) - h_\times(x^\mu_p,\hat{\Omega})\right]\Big\}$$

$\hat{p}$ = unit vector pointing to pulsar

$\hat{\Omega}$ = unit vector pointing to GW source

e^+_{ij}, $e^\times_{ij}$ = polarization tensors

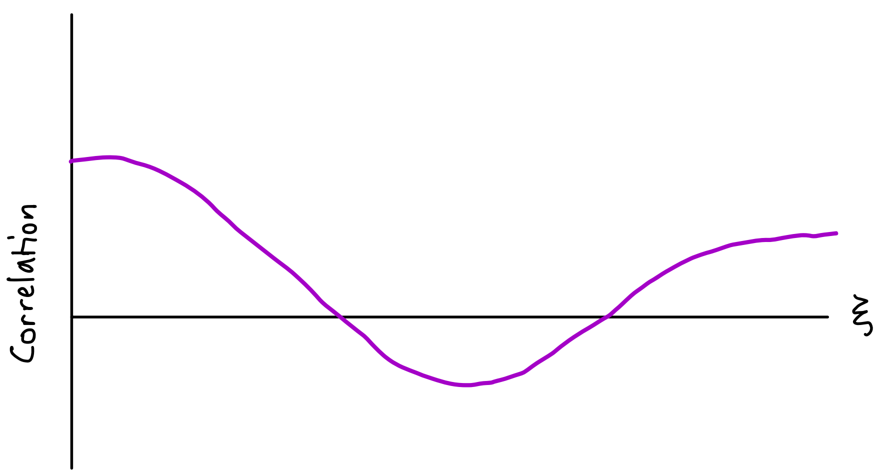
Pulsar Timing Arrays

Gravitational waves induce a redshift in the pulses of pulsars.

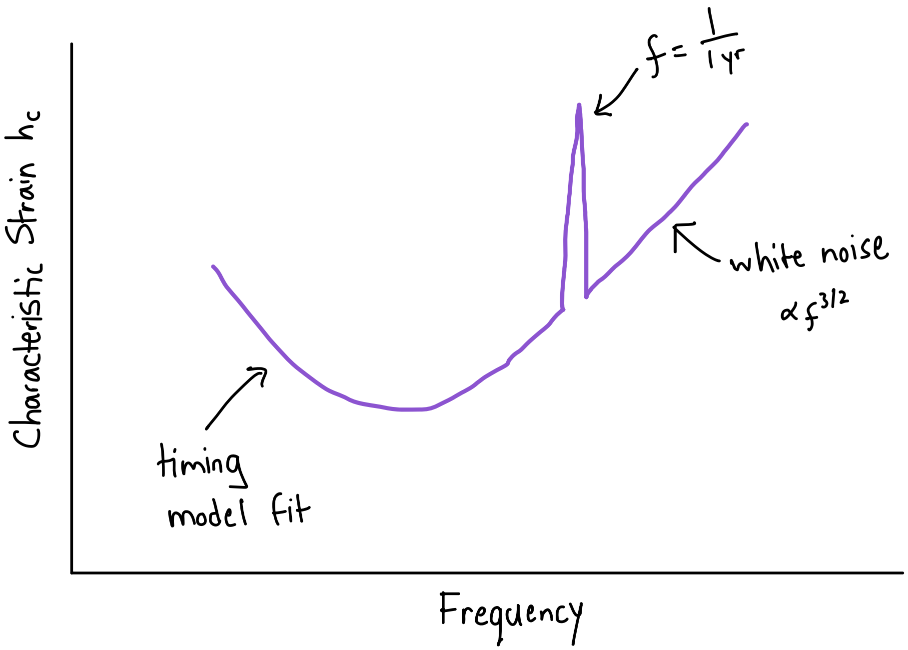
We search for gravitational waves by searching the pulsar timing residuals:

$$r_{GW}(t) = \int_{t_0}^{t} dt' \, \frac{\nu_p - \nu_\varepsilon}{\nu_\varepsilon} = \underbrace{\frac{1}{2} \frac{\hat{p}^i \cdot \hat{p}^j}{1 + \hat{\Omega}\cdot\hat{p}} \sum_{A=+,\times} e^A_{ij}(\hat{\Omega})}_{\text{response of a pulsar to a GW source}} \underbrace{\int_{t_0}^{t} dt' \left[h_A(x^\mu_\varepsilon) - h_A(x^\mu_p) \right]}_{\text{GW signal}}$$

For an ensemble of sources, we integrate over a population of sources distributed across the sky. Doing this gives the Hellings-Downs curve:



Pulsar timing arrays are sensitive to GWs from $f \sim 1-100$ nHz.



strain $h(t) \sim \frac{\Delta L}{L}$ dimensionless

strain noise PSD $S_h(f)$ units $\frac{1}{Hz}$

$$S_h(f)\,df = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left| h_{(f, f+df)}(t) \right|^2$$

characteristic strain $h_c(f) \sim f\,\tilde{h}(f)$

dimensionless in Fourier space

$$S_h(f) \sim \frac{h_c^2(f)}{f}$$

PTAs often describe things in terms of the induced residuals:

$$S_r(f) \sim \frac{1}{f^2} S_h(f) \sim \frac{h_c^2(f)}{f^3}$$

residual PSD

For white noise, $S_r(f) \sim f^0 \Rightarrow h_c(f) \sim f^{3/2}$.