June 18th 2026

The astrophysical interpretation of Pulsar Timing Array measurements

IPTA Student Week 2026

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Supermassive Black Holes

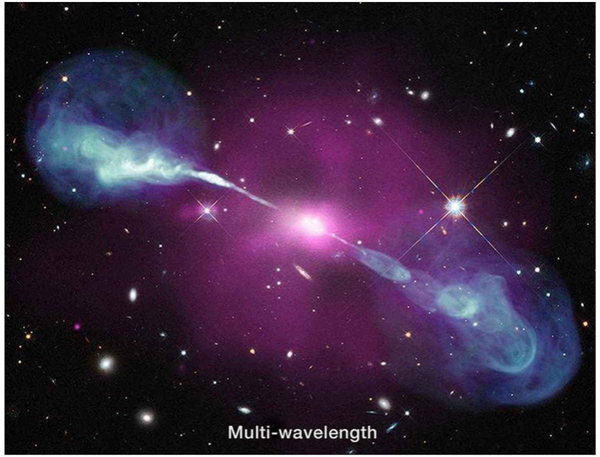
→ The most massive compact objects in our universe $M > 10^6 M_\odot$

Different observational evidence

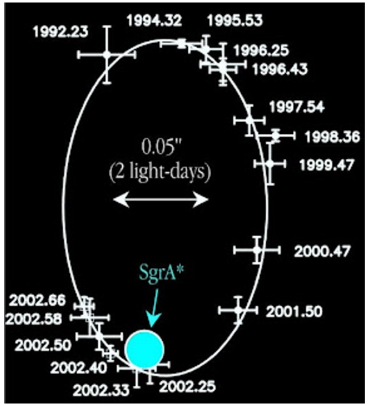
→ Active Galactic Nuclei (AGN), Quasars, ...

→ Stellar dynamics in galaxy cores

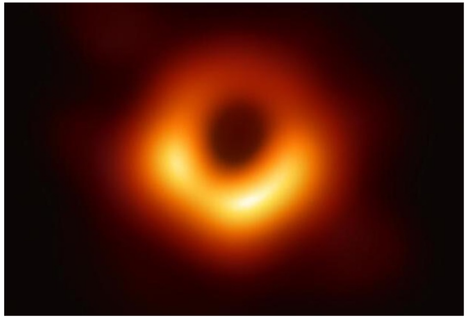
→ “Direct” observation



Credits: science.nasa.gov



Adapted from Schödel et al, Nature (2002)

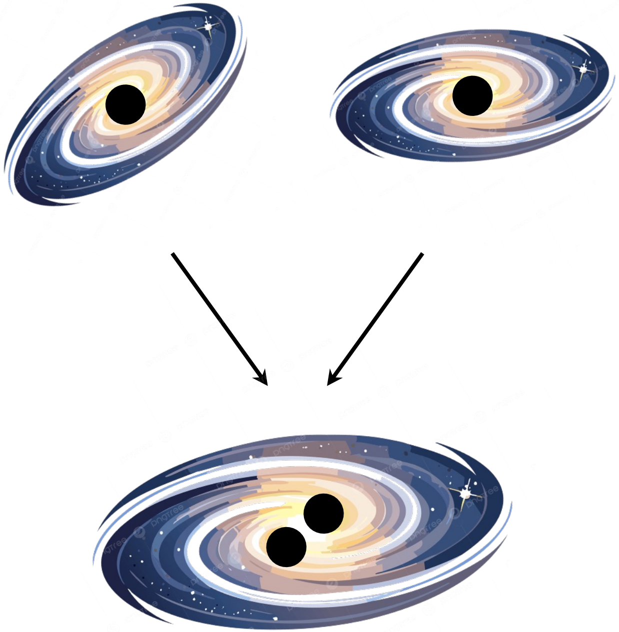


Credits: Event Horizon Telescope

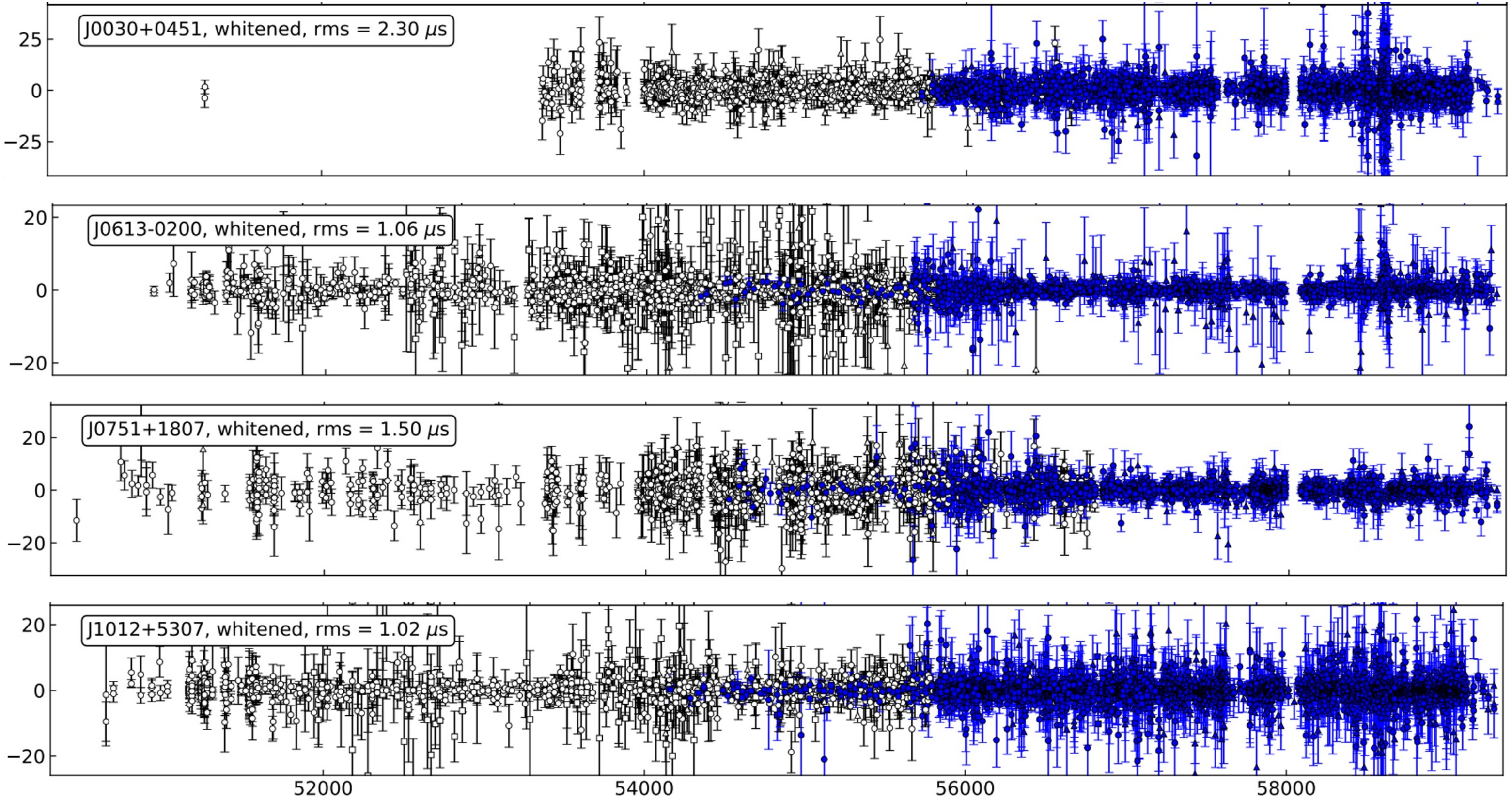
Supermassive Black Hole Binaries

A consequence of the hierarchical formation of galaxies

Credits: ESA



What signal will they leave in the PTA data?



EPTA (2023)

The Bayesian approach

What is the astrophysical model?

Ingredients:

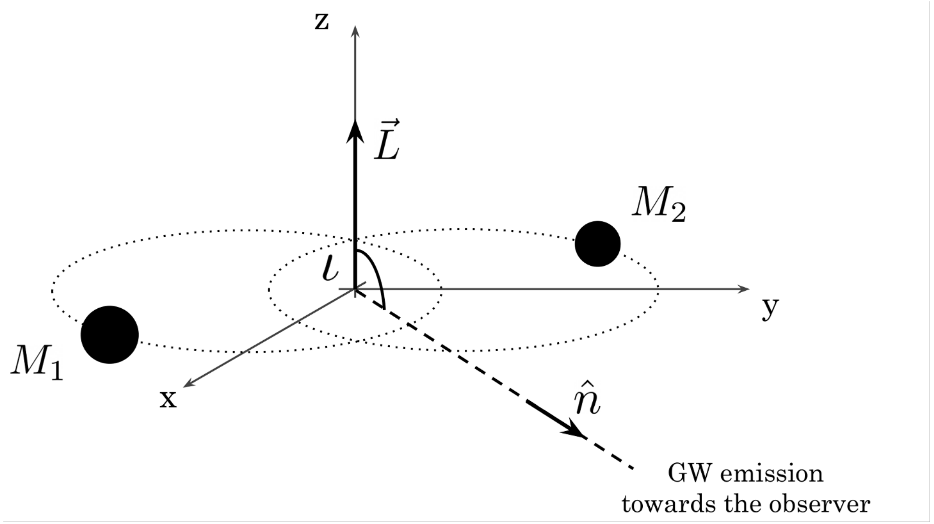
- a likelihood $p(d \mid \vec{\theta}, \mathcal{M}, \mathcal{I})$
- a prior $p(\vec{\theta} \mid \mathcal{M}, \mathcal{I})$

Constraining models: we can derive posterior distributions $p(\vec{\theta} \mid d, \mathcal{M}, \mathcal{I})$

Model comparison: we can use the evidence of the model $p(d \mid \mathcal{M}, \mathcal{I})$

$$\mathcal{M}_\mathrm{c} = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}}$$

The "true" astrophysical model



The signal from each binary can be approximated by an analytical waveform which depends on the binary parameters

$$\hat{\Omega}, d_L, \cos\iota, \psi, \mathcal{M}_c,$$
$$q, e_0, f_\mathrm{orb}^{(0)}, \phi_\mathrm{orb}^{(0)}$$

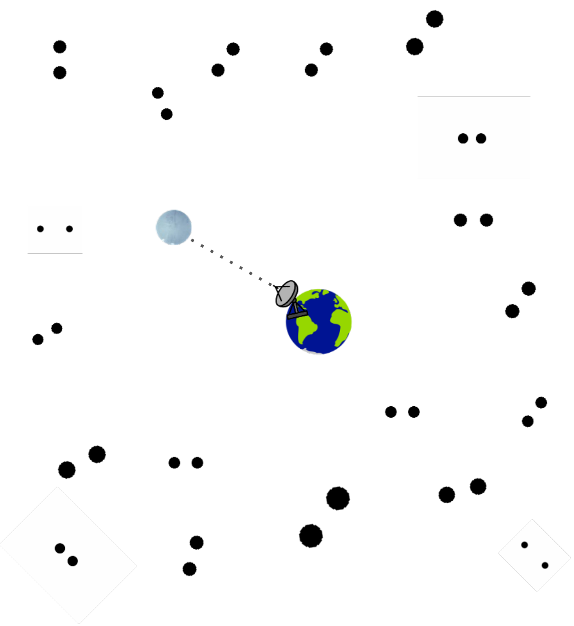
and an assumed orbital dynamics

The binary induces timing residuals in each pulsar

$$R_\mathrm{GW}^{(a)}(t) = \sum_{A\in\{+,\times\}} F_a^A \int^t \mathrm{d}t' \left[h_A(t') - h_A(t' - \tau_a)\right]$$

More details in Boris Goncharov lecture!

The "true" astrophysical model



The total timing residuals signal of the astrophysical background is then

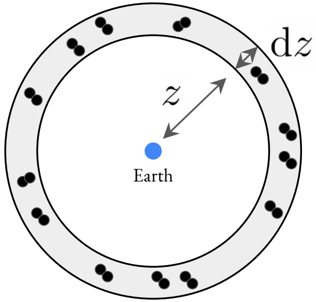
$$R_{\mathrm{GW}}^{(a)}(t) = \sum_{i=1}^{N_{\mathrm{b}}} R_{\mathrm{b}}^{(a)}(t\,;\vec{\theta}_i)$$

In theory, we can then search for the parameters of each binary

$$\vec{\theta} = \bigcup_{i=1}^{N_{\mathrm{b}}} \vec{\theta}_i$$

PROBLEM? The large number of binaries emitting in the PTA band makes this approach impractical

The expected number of inspiralling SMBHBs



How many binaries are emitting in a given log-frequency bin?

$$\Delta N(f) \simeq \int dz \left.\frac{dN}{dz\, d\ln f_{\rm orb}}\right|_{f_{\rm orb}=\frac{(1+z)f}{2}} \Delta \ln f$$

$$= \int dz \frac{dV_c}{dz} \times \left[\frac{dN}{dV_c d\tau_c} \times \frac{d\tau_c}{d\ln f_{\rm orb}}\right]_{f_{\rm orb}=\frac{(1+z)f}{2}} \Delta \ln f$$

$\dfrac{dN}{dV_c d\tau_c}$ Comoving density of inspiralling binaries that reach a given time to coalescence, per unit time

$\dfrac{\mathrm{d}\tau_c}{\mathrm{d}\ln f_{\rm orb}}$ Residence time of binaries per log-frequency bin

The expected number of inspiralling SMBHBs

$$\Delta N(f) = \int dz \frac{dV_c}{dz} \times \left[\frac{dN}{dV_c d\tau_c} \times \frac{d\tau_c}{d\ln f_{\rm orb}} \right]_{f_{\rm orb} = \frac{(1+z)f}{2}} \Delta \ln f$$

$$\frac{dN}{dV_c d\tau_c}(\tau_c) = \frac{dN}{dV_c d\tau_c}(\tau_c = 0)$$

If we assume a steady rate of merging SMBHBs, $\dfrac{dN}{dV_c d\tau_c}$ reduces to the comoving merger rate of SMBHBs

In addition, for a population of circular binaries, following Peters & Matthews, the GW radiation implies:

$$\frac{\mathrm{d}\tau_\mathrm{c}}{\mathrm{d}\ln f_{\rm orb}} = \frac{5}{96} \left(\frac{G\mathcal{M}_\mathrm{c}}{c^3} \right)^{-5/3} (2\pi f_{\rm orb})^{-8/3}$$

Neglecting the redshift dependence in the integral and considering a population with a characteristic chirp mass, we have:

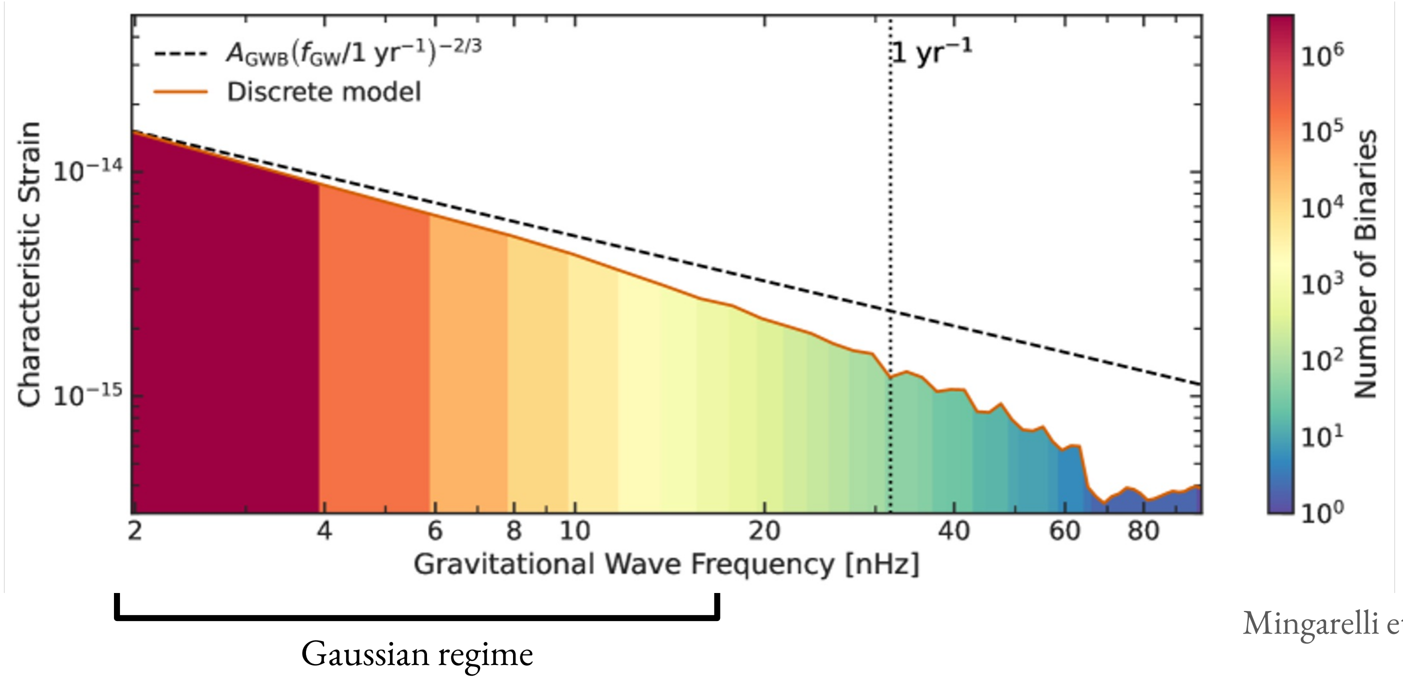
$$\Delta N(f) \approx V_c(z \leq 1) \times \dot{n}(z_\star)(1+z_\star)^{-8/3} \times \tau_c(f)$$

$\Delta f = 1/T_{\rm obs}$

$$\sim 2 \times 10^4 \left(\frac{\dot{n}(z_\star)}{10^{-2}\,\mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}} \right) \left(\frac{\mathcal{M}_{c,\star}}{10^9\, M_\odot} \right)^{-5/3} \left(\frac{f}{10\,\mathrm{nHz}} \right)^{-11/3} \left(\frac{10\,\mathrm{yr}}{T_{\rm obs}} \right)$$

The expected number of inspiralling SMBHBs

$$\Delta N(f) \sim 2 \times 10^4 \left(\frac{\dot{n}(z_\star)}{10^{-2}\,\mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}} \right) \left(\frac{\mathcal{M}_{c,\star}}{10^9\,M_\odot} \right)^{-5/3} \left(\frac{f}{10\,\mathrm{nHz}} \right)^{-11/3} \left(\frac{10\,\mathrm{yr}}{T_{\mathrm{obs}}} \right)$$



Mingarelli et al. (2026)

The Gaussian unpolarized isotropic and stationary model

- Thanks to this high number of identically distributed sources, we can model the signal as a Gaussian stochastic process (Central Limit Theorem)
- Supposing these thousands of binaries are isotropically distributed over the sky (neglecting galaxy clustering, large scale structures, ...), the GW power is isotropic
- Neglecting the orbital frequency evolution of SMBHBs over the observation time scale, we can assume the process is stationary

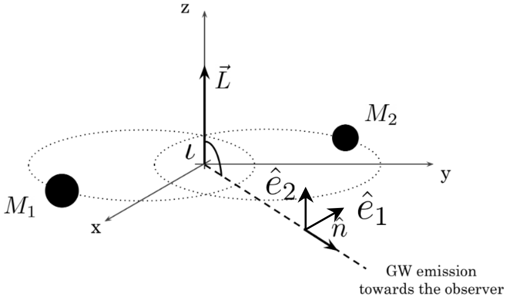
$$\begin{aligned}\frac{\mathrm{d}f_{\mathrm{GW}}}{\mathrm{d}t} &= \pi^{8/3}\frac{96}{5}\left(\frac{G\mathcal{M}_{\mathrm{c}}}{c^3}\right)^{5/3} f_{\mathrm{GW}}^{11/3} \\ &\simeq 8.5\times 10^{-5}\left(\frac{\mathcal{M}_{\mathrm{c}}}{10^9\,\mathrm{M}_{\odot}}\right)^{5/3}\left(\frac{f_{\mathrm{GW}}}{10\,\mathrm{nHz}}\right)^{11/3}\ \mathrm{nHz/year}\end{aligned}$$

The Gaussian unpolarized isotropic and stationary model

➢ The GW signal from a single SMBHB is *a priori* elliptically polarized

$$h_+(f) \propto A_+ e^{i\phi_0} \qquad h_\times(f) \propto i\, A_\times e^{i\phi_0}$$

$$A_+ \equiv \frac{h_0}{2}(1+\cos^2\iota) \qquad A_\times \equiv h_0 \cos\iota$$



➢ Considering the superposition of signals coming from a given direction at a given GW frequency, and fixing the polarization frame, we have

$$h_1(f) = \sum_k \left(h_+^{(k)} \cos 2\psi_k - h_\times^{(k)} \sin 2\psi_k \right) \qquad h_2(f) = \sum_k \left(h_+^{(k)} \sin 2\psi_k + h_\times^{(k)} \cos 2\psi_k \right)$$

➢ Exercise: Show that the resulting background is unpolarized, i.e., $\begin{cases} \langle h_1^* \, h_1 \rangle_{\cos\iota,\,\psi} = \langle h_2^* \, h_2 \rangle_{\cos\iota,\,\psi} \\ \langle h_1^* \, h_2 \rangle_{\cos\iota,\,\psi} = 0 \end{cases}$

Bonus: What are the minimal conditions to impose on $\cos\iota$ *and* ψ *distributions in order to have an unpolarized background?*

The Gaussian unpolarized isotropic and stationary model

- All of this justifies the plane wave expansion to model the astrophysical GW background

$$h_{ij}^{(\mathrm{TT})}(t,\vec{x}) = \sum_{A\in\{+,\times\}} \int_{S^2} \mathrm{d}^2\Omega_{\hat{n}} \int_{-\infty}^{+\infty} \mathrm{d}f \;\; \tilde{h}_A(f,\hat{n}) e_{ij}^A(\hat{n}) e^{i2\pi f\left(t-\frac{\hat{n}\cdot\vec{x}}{c}\right)}$$

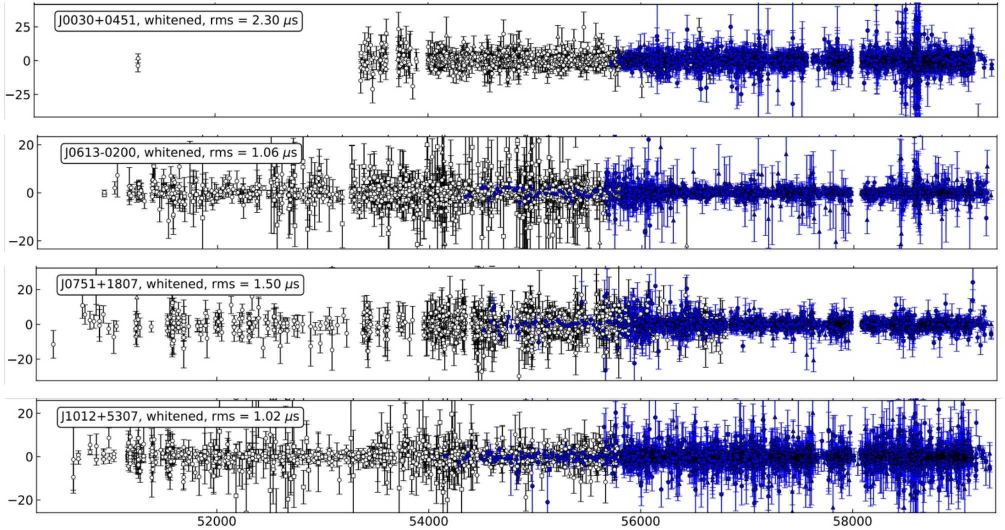
where the strain Fourier modes are Gaussian random variables which verify

$$\begin{aligned}\left\langle \tilde{h}_A(f,\hat{n})\tilde{h}_{A'}\left(f',\hat{n}'\right)^*\right\rangle &= S_{\mathrm{h}}(f)\frac{\delta\left(f-f'\right)}{2}\frac{\delta^{(3)}\left(\hat{n}-\hat{n}'\right)}{4\pi}\frac{\delta_{AA'}}{2} \\ &= \frac{1}{16\pi}S_{\mathrm{h}}(f)\delta\left(f-f'\right)\delta^{(3)}\left(\hat{n}-\hat{n}'\right)\delta_{AA'}\end{aligned}$$

- $S_{\mathrm{h}}(f)$ is the (one-sided) strain Power Spectral Density (PSD) of the background

From strain to timing residuals: the GW background case

$$S_{\mathrm{h}}(f) \quad \xleftrightarrow{\;?\;} $$



⋮

From strain to timing residuals: the GW background case

$$h_{ij}^{(\mathrm{TT})}(t,\vec{x}) = \sum_{A\in\{+,\times\}} \int_{S^2} \mathrm{d}^2\Omega_{\hat{n}} \int_{-\infty}^{+\infty} \mathrm{d}f\ \tilde{h}_A(f,\hat{n}) e_{ij}^A(\hat{n}) e^{i2\pi f\left(t-\frac{\hat{n}\cdot\vec{x}}{c}\right)}$$

$$\left\langle \tilde{h}_A(f,\hat{n})\tilde{h}_{A'}(f',\hat{n}')^* \right\rangle = S_\mathrm{h}(f)\frac{\delta(f-f')}{2}\frac{\delta^{(3)}(\hat{n}-\hat{n}')}{4\pi}\frac{\delta_{AA'}}{2}$$
$$= \frac{1}{16\pi} S_\mathrm{h}(f)\delta(f-f')\,\delta^{(3)}(\hat{n}-\hat{n}')\,\delta_{AA'}$$

$$\longrightarrow \quad R_{\mathrm{GW}}^{(a)}(t) = \sum_{A\in\{+,\times\}} F_a^A \int^t \mathrm{d}t' \left[h_A(t') - h_A(t'-\tau_a)\right]$$

- This implies for the two-point function of the GW background:

$$\left\langle \tilde{R}_{\mathrm{GWB}}^{(a)}(f)\tilde{R}_{\mathrm{GWB}}^{(b)}(f')^* \right\rangle = \frac{1}{2}\bar{\Gamma}_{ab}^{(\mathrm{HD})}\frac{S_\mathrm{h}(f)}{12\pi^2 f^2}\delta(f-f')$$

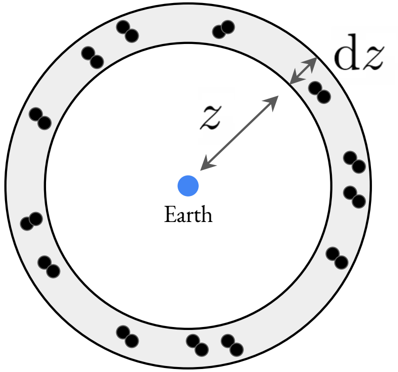
$$\bar{\Gamma}_{ab}^{(\mathrm{HD})} = \frac{1}{2}\delta_{ab} + \frac{1}{2} + \frac{3}{2}\xi_{ab}\left[\ln(\xi_{ab}) - \frac{1}{6}\right]$$

- Measuring the timing residuals PSD of the background is therefore an indirect measurement of $S_\mathrm{h}(f)$

$$S_\mathrm{R}^{(\mathrm{GWB})}(f) = \frac{S_\mathrm{h}(f)}{12\pi^2 f^2}$$

What can we learn from inferring the signal power spectral density?

➢ The background power spectrum is usually quantified in terms of characteristic strain $h_c^2(f) \equiv f S_h(f)$



$$h_c^2(f) = \int \mathrm{d}z \mathrm{d}\vec{\xi} \mathrm{d}\ln f_{\mathrm{orb}} \frac{\mathrm{d}^3 N_i}{\mathrm{d}z \mathrm{d}\vec{\xi} \mathrm{d}\ln f_{\mathrm{orb}}} h_{c,1}^2(f; z, \vec{\xi}, f_{\mathrm{orb}})$$

➢ As shown previously, the distribution of inspiralling binaries is directly linked to the comoving merger rate density

$$\frac{\mathrm{d}^3 N_i}{\mathrm{d}z \mathrm{d}\vec{\xi} \mathrm{d}\ln f_{\mathrm{orb}}} = \frac{\mathrm{d}^2 n}{\mathrm{d}\vec{\xi} \mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{d}\tau_c} \frac{\mathrm{d}\tau_c}{\mathrm{d}\ln f_{\mathrm{orb}}} \frac{\mathrm{d}V_c}{\mathrm{d}z}$$

$$\vec{\xi} = (M_1, q, e_0, \sigma_{\mathrm{star}}, \ldots)$$

Merger rate density of SMBHBs

Dynamics of SMBHBs

What can we learn from inferring the signal power spectral density?

$$S_{\mathrm{R}}^{(\mathrm{GWB})}(f) = \frac{h_{\mathrm{c}}^2(f)}{12\pi^2 f^3}$$

$$h_{\mathrm{c}}^2(f) = \int \mathrm{d}z \mathrm{d}\vec{\xi} \mathrm{d}\ln f_{\mathrm{orb}} \frac{\mathrm{d}^3 N_i}{\mathrm{d}z \mathrm{d}\vec{\xi} \mathrm{d}\ln f_{\mathrm{orb}}} h_{\mathrm{c},1}^2(f; z, \vec{\xi}, f_{\mathrm{orb}})$$

$$\frac{\mathrm{d}^3 N_i}{\mathrm{d}z \mathrm{d}\vec{\xi} \mathrm{d}\ln f_{\mathrm{orb}}} = \frac{\mathrm{d}^2 n}{\mathrm{d}\vec{\xi} \mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{d}\tau_{\mathrm{c}}} \frac{\mathrm{d}\tau_{\mathrm{c}}}{\mathrm{d}\ln f_{\mathrm{orb}}} \frac{\mathrm{d}V_{\mathrm{c}}}{\mathrm{d}z}$$

$$\cdot h_{\mathrm{c},1}^2(f; z, \vec{\xi}, f_{\mathrm{orb}})$$

Merger rate density of SMBHBs

Dynamics of SMBHBs

GW radiation
Scattering with stars
Interaction with gas
...

GW radiation spectrum

Eccentricity

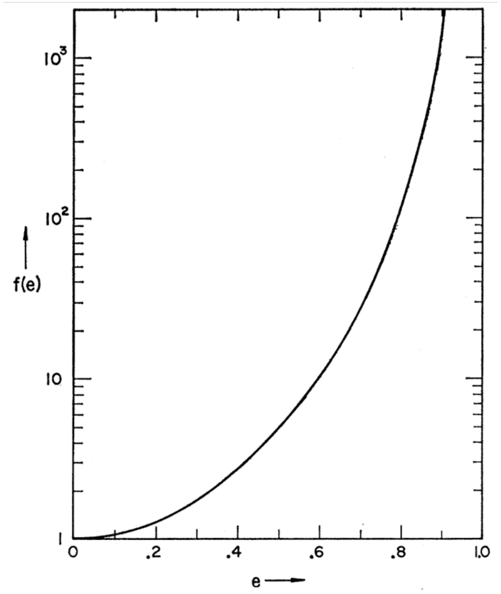
Example: the impact of orbital eccentricity

[Enoki & Nagashima (2006)]

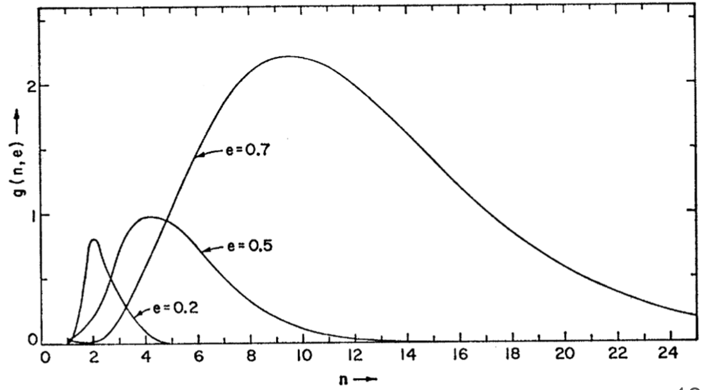
- Orbital eccentricity both
  - Boosts the total GW radiation, accelerating the orbital hardening

$$\frac{\mathrm{d}\tau_c}{\mathrm{d}f_{\mathrm{orb}}} = \frac{1}{F(e)}\frac{\mathrm{d}\tau_c^{(\mathrm{circ})}}{\mathrm{d}f_{\mathrm{orb}}}$$

  - Shifts the GW radiation towards higher orbital harmonics



[Peters & Matthews (1963)]



$$h^2_{\mathrm{c},1}(f) = \frac{4G}{\pi c^2 f^2}\frac{\mathrm{d}\rho^{(1)}_{\mathrm{GW}}}{\mathrm{d}\ln f}(f)$$

$$\frac{\mathrm{d}\rho^{(1)}_{\mathrm{GW}}}{\mathrm{d}\ln f}(f;\vec{\xi}) = \frac{L^{(\mathrm{circ})}_{\mathrm{GW}}}{c4\pi d_{\mathrm{L}}^2}\sum_m g(m,e) f_{\mathrm{s}}\delta\left(f_{\mathrm{s}} - m f_{\mathrm{orb}}\right)$$

$$L^{(\mathrm{circ})}_{\mathrm{GW}}(\vec{\xi}) = \frac{32}{5}\frac{G^{7/3}}{c^5}\mathcal{M}_{\mathrm{c}}^{10/3}\left(2\pi f_{\mathrm{orb}}\right)^{10/3}$$

$$f_{\mathrm{s}} \equiv (1+z)f$$

Example: the impact of orbital eccentricity

[Enoki & Nagashima (2006)]

$$h_{\rm c}^2(f) = \int {\rm d}z{\rm d}\vec{\xi}{\rm d}f_{\rm orb}\frac{{\rm d}^3 N_{\rm i}}{{\rm d}z{\rm d}\vec{\xi}{\rm d}f_{\rm orb}} h_{{\rm c},1}^2(f;z,\vec{\xi},f_{\rm orb})$$

Injecting the previous spectrum for eccentric binaries, and using

$$\frac{{\rm d}^3 N_{\rm i}}{{\rm d}z{\rm d}\vec{\xi}{\rm d}f_{\rm orb}} = \frac{{\rm d}^2 n}{{\rm d}\vec{\xi}{\rm d}z}\frac{{\rm d}\tau_{\rm c}}{{\rm d}f_{\rm orb}}\frac{{\rm d}V_{\rm c}}{{\rm d}z}\frac{{\rm d}z}{{\rm d}\tau_{\rm c}} = \frac{{\rm d}^2 n}{{\rm d}\vec{\xi}{\rm d}z}\frac{{\rm d}\tau_{\rm c}}{{\rm d}f_{\rm orb}} \times \frac{4\pi d_{\rm L}^2 c}{1+z}$$

One obtains,

$$h_{\rm c}^2(f) = \frac{4G}{\pi c^2 f^2}\int {\rm d}z{\rm d}\vec{\xi}{\rm d}f_{\rm orb}\frac{{\rm d}^2 n}{{\rm d}\vec{\xi}{\rm d}z} L_{\rm GW}^{\rm (circ)}\frac{{\rm d}\tau_{\rm c}}{{\rm d}f_{\rm orb}}\sum_m g[m,e(f_{\rm orb})]\frac{f}{m}\delta\left(\frac{(1+z)f}{m} - f_{\rm orb}\right)$$

$$= \frac{4G}{\pi c^2 f}\int {\rm d}z{\rm d}\vec{\xi}\frac{{\rm d}^2 n}{{\rm d}\vec{\xi}{\rm d}z}\sum_m \left\{ L_{\rm GW}^{\rm (circ)}\frac{{\rm d}\tau_{\rm c}}{{\rm d}f_{\rm orb}}\frac{g[m,e(f_{\rm orb})]}{m}\right\}_{f_{\rm orb}=\frac{(1+z)f}{m}}$$

$$= \frac{4\pi c^3}{3}\int {\rm d}z{\rm d}\vec{\xi}\frac{{\rm d}^2 n}{{\rm d}\vec{\xi}{\rm d}z}(1+z)^{-1/3}\left(\frac{G\mathcal{M}_{\rm c}}{c^3}\right)^{5/3}(\pi f)^{-4/3}\times\Phi(f)$$

Example: the impact of orbital eccentricity

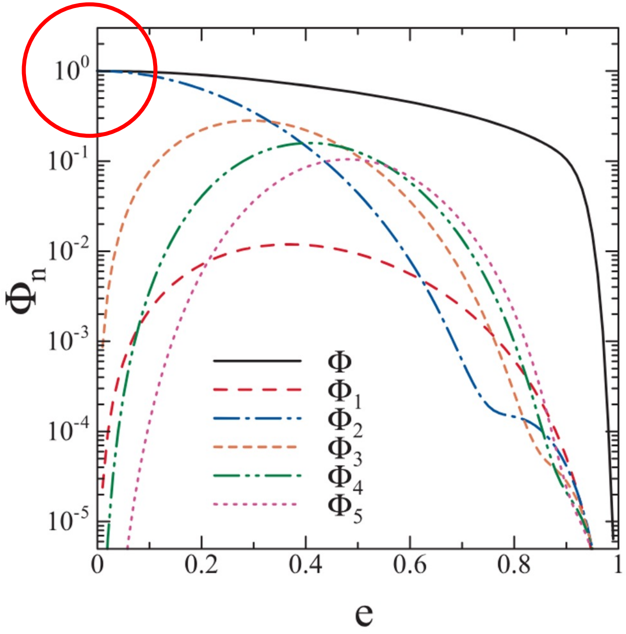
[Enoki & Nagashima (2006)]

$$h_c^2(f) = \frac{4\pi c^3}{3} \int \mathrm{d}z \mathrm{d}\vec{\xi} \frac{\mathrm{d}^2 n}{\mathrm{d}\vec{\xi}\mathrm{d}z} (1+z)^{-1/3} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{5/3} (\pi f)^{-4/3} \times \Phi(f)$$

with,

$$\Phi(f) = \sum_{m \geq 1} \left(\frac{2}{m} \right)^{2/3} \left. \frac{g(m,e)}{F(e)} \right|_{e\left[\frac{(1+z)f}{m}\right]}$$

➢ In the circular case, we recover the expected power law



Example: the impact of orbital eccentricity

[Enoki & Nagashima (2006)]

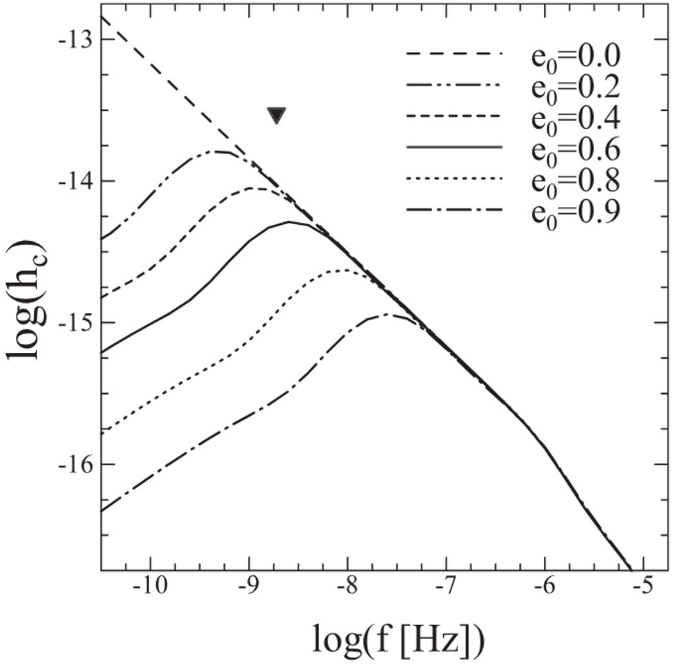
$$h_c^2(f) = \frac{4\pi c^3}{3} \int \mathrm{d}z \mathrm{d}\vec{\xi} \frac{\mathrm{d}^2 n}{\mathrm{d}\vec{\xi} \mathrm{d}z} (1+z)^{-1/3} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{5/3} (\pi f)^{-4/3} \times \Phi(f)$$

with,

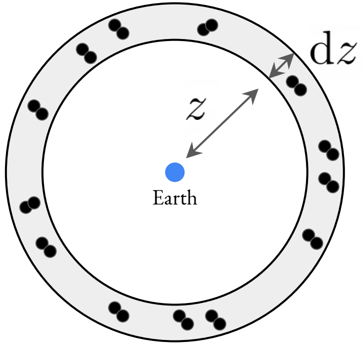
$$\Phi(f) = \sum_{m \geq 1} \left(\frac{2}{m} \right)^{2/3} \frac{g(m,e)}{F(e)} \Bigg|_{e\left[\frac{(1+z)f}{m}\right]}$$

- ➢ In the circular case, we recover the expected power law
- ➢ If the population is eccentric, the Φ function modulates the GW background spectrum at low frequency

→ Measuring/Constraining the bending of the GW background PSD at low frequency can therefore constrain the eccentricity distribution of SMBHBs



What about the comoving merger rate function?



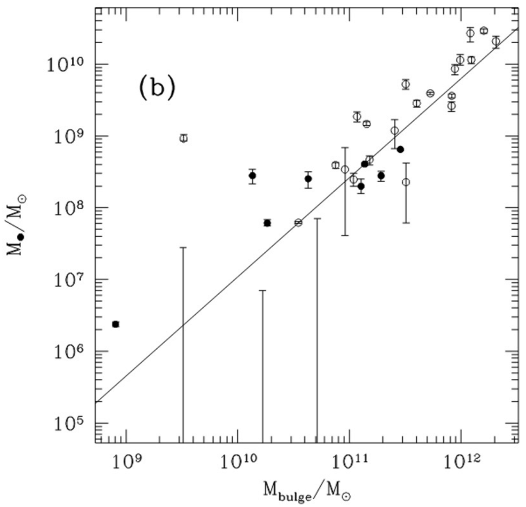
$$\frac{\mathrm{d}^2 n}{\mathrm{d}\vec{\xi}\,\mathrm{d}z}$$ Distribution of merging SMBHBs per unit comoving volume

- ➢ Estimating this quantity is a multi-scale problem since SMBHB mergers follow galaxy mergers that, themselves, follow dark matter halo mergers
- ➢ Methods that estimate the comoving merger density of SMBHBs emitting in the PTA band therefore need to model
  - The assembly of massive galaxies across cosmic time down to z ~ 0
  - The dynamics of SMBHs from galaxy pairing (kpc scales) down to their coalescence

$$R_\mathrm{s} \sim 0.1\ \mathrm{mpc} \times \frac{M_\mathrm{BH}}{10^9 M_\odot}$$

Three main methods

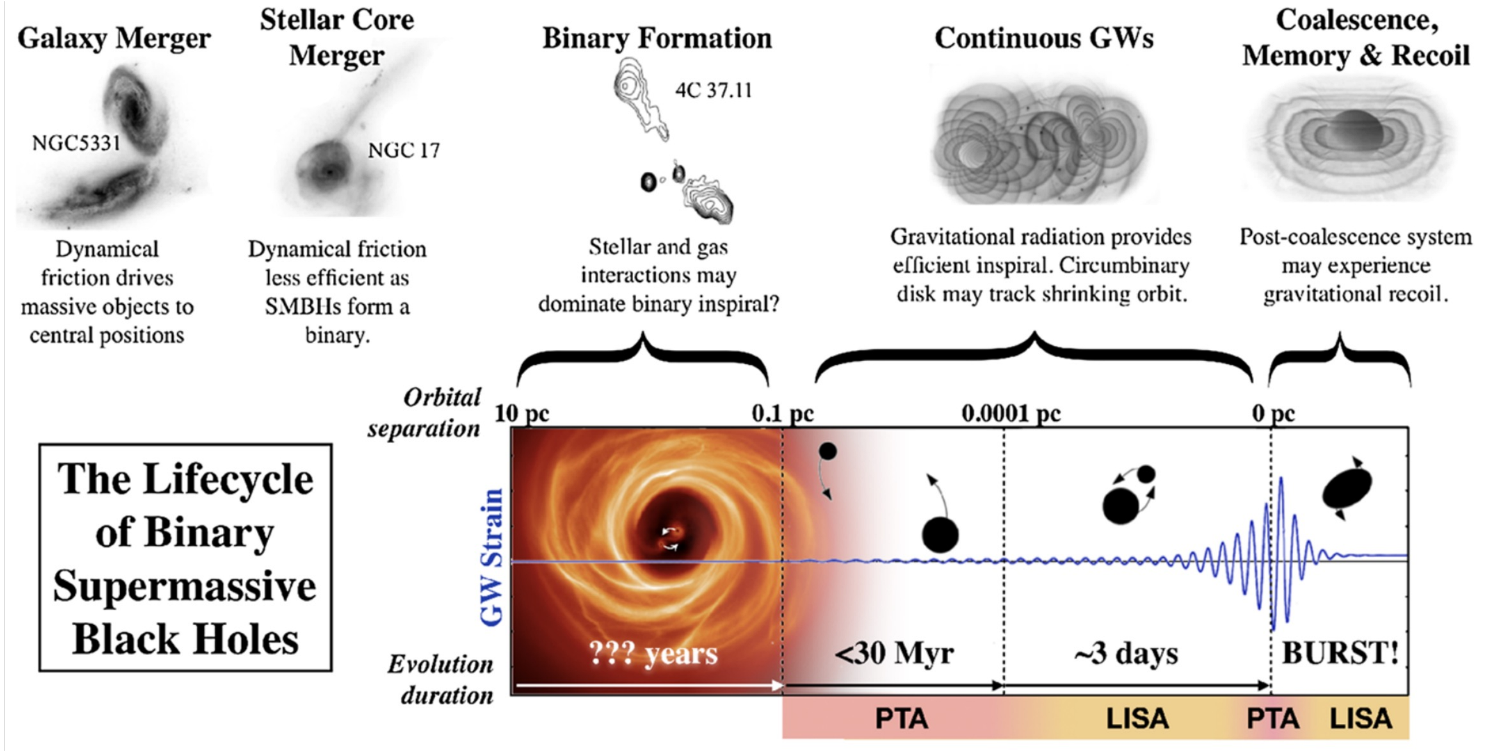
- ➢ empirical models based on observations [Sesana 2013, Rosado et al. 2015, Ravi et al. 2015, ...]
  - ○ galaxy stellar mass function, pair fraction, scaling relations, ...
- ➢ semi-analytical models [Henriques et al. 2015, Barausse 2012, ...]
  - ○ based on dark matter halo merger tree
  - ○ baryonic physics is added on top along with MBHs evolution
- ➢ cosmological hydrodynamical simulations [Illustris, Horizon-AGN, Eagle, ...]
  - ○ consistent co-evolution of dark matter, gas, stellar particles and MBHs in a large comoving volume (box)



Magorrian et al. 1998

The standard dynamical model after galaxy pairing

- This model was initially proposed by Begelman, Blandford & Rees (1980)



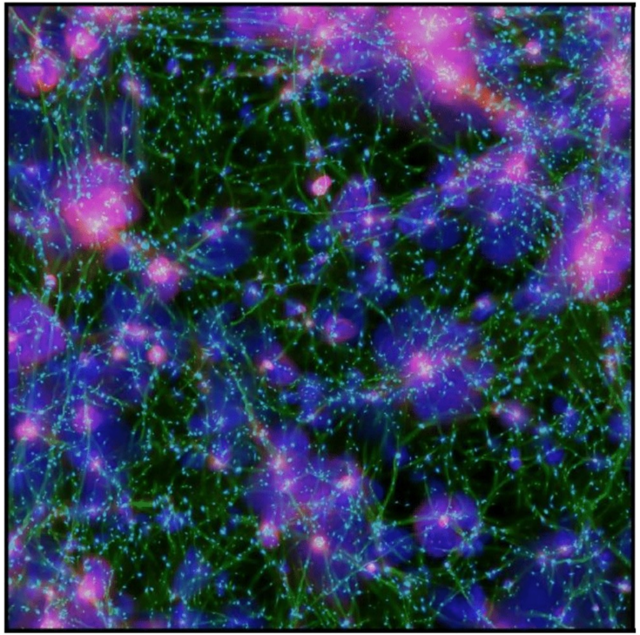
Credits: S. Burke-Spolaor, et al. (2019)

Example: Hydrodynamical cosmological simulations

- MBHs are seeded at high redshift (heavy seed: $M_{\rm seed} \sim 10^5 M_\odot$)
- Grow through
  - gas accretion (given local gas properties)

$$\dot{M}_{\rm BH} = 4\pi\alpha_{\rm b} G^2 M_{\rm BH}^2 \bar{\rho}/(\bar{c}_s^2 + \bar{u}^2)^{3/2}$$

  - mergers
- Problem: such large volume simulations usually have resolution of the order of kpc → still far from merger!
- To pass from numerical merger properties $\left(z_j^{\rm (num)}, \vec{\xi}_j^{\rm (num)}\right)$ to properties of the SMBHB at the "real" merger, dynamical timescales are usually used, that depends on the galaxy properties (stellar mass, stellar velocity dispersion, …)



Dubois et al. (2014)

$$t_{\rm merger} = t_{\rm num} + t_{\rm DF} + t_{\rm hardening} \longrightarrow \frac{{\rm d}^2 n}{{\rm d}z{\rm d}\vec{\xi}} \simeq \frac{1}{V_{\rm sim}} \sum_j \delta(z - z_j)\delta(\vec{\xi} - \vec{\xi}_j)$$

Recap of this (and previous) lectures

	Pulsar a	Pulsar b	Pulsar c	
Pulsar a	$C_{MN} + C_{iRN} + C_{DMv} + C_{GWB}$	C_{GWB}	C_{GWB}	$\cdots$
Pulsar b	C_{GWB}	$C_{MN} + C_{iRN} + C_{DMv} + C_{GWB}$	C_{GWB}	
Pulsar c	C_{GWB}	C_{GWB}	$C_{MN} + C_{iRN} + C_{DMv} + C_{GWB}$	
	$\vdots$			$\ddots$

→ By cross-correlating pulsar timing residuals, we can estimate

$$S_{\mathrm{R}}^{(\mathrm{GWB})}(f)$$

→ At low frequency (< 20 nHz), an astrophysical GW background can be modeled as an isotropic, stationary and un-polarized Gaussian process, such that

$$S_{\mathrm{R}}^{(\mathrm{GWB})}(f) = \frac{S_{\mathrm{h}}(f)}{12\pi^2 f^2}$$

→ The strain PSD is complex to model and reflects the co-evolution of massive galaxies and the SMBHs they host. Its amplitude and spectral shape depends both on

1. The comoving merger density of binaries
2. The dynamics of SMBHBs (eccentricity, stellar hardening, ...)

Can we go beyond? An exciting field of research!

→ including the search for individual sources on top of the background signal (*See the next lecture!*)

→ we can go beyond the assumptions used in this lecture, searching for

- → anisotropies
- → non-stationarities (eccentricity, high frequency regime)
- → non-gaussianities? (high frequency regime)

→ …

Questions?

Recall $$\frac{\mathrm{d}^3 N_i}{\mathrm{d}z\mathrm{d}\vec{\xi}\mathrm{d}\ln f_{\mathrm{orb}}} = \frac{\mathrm{d}^2 n}{\mathrm{d}\vec{\xi}\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\tau_{\mathrm{c}}}\frac{\mathrm{d}\tau_{\mathrm{c}}}{\mathrm{d}\ln f_{\mathrm{orb}}}\frac{\mathrm{d}V_{\mathrm{c}}}{\mathrm{d}z}$$

Example: Hydrodynamical cosmological simulations

$$\frac{\mathrm{d}^2 n}{\mathrm{d}z\mathrm{d}\vec{\xi}} \simeq \frac{1}{V_{\mathrm{sim}}}\sum_j \delta(z - z_j)\delta(\vec{\xi} - \vec{\xi}_j)$$

Computing the average GW background spectrum

Computing the GW background for different universe realizations

We can inject this expression in the Phinney formula (see Alberto's talk)

$$h_{\mathrm{c}}^2(f) = \frac{4G}{\pi c^2}\frac{1}{f^2}\int \mathrm{d}z\,\mathrm{d}\vec{\xi}\,\frac{\mathrm{d}^2 n}{\mathrm{d}z\mathrm{d}\vec{\xi}}\frac{1}{1+z}\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\ln f_{\mathrm{s}}}(f_{\mathrm{s}})$$

We can sample the distribution of inspiralling SMBHBs over a discretized orbital frequency grid

$$\begin{aligned}
\langle N_{\mathrm{i}}^{(j)}\rangle(f_{\mathrm{orb}}) &= \int \mathrm{d}z\mathrm{d}\vec{\xi}\int_{\ln f_{\mathrm{orb}}-\Delta\ln f_{\mathrm{orb}}/2}^{\ln f_{\mathrm{orb}}+\Delta\ln f_{\mathrm{orb}}/2}\mathrm{d}\ln f_{\mathrm{orb}}\frac{\mathrm{d}^2 n^{(j)}}{\mathrm{d}\vec{\xi}\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\tau_{\mathrm{c}}}\frac{\mathrm{d}\tau_{\mathrm{c}}}{\mathrm{d}\ln f_{\mathrm{orb}}}\frac{\mathrm{d}V_{\mathrm{c}}}{\mathrm{d}z}\\
&= \int \mathrm{d}z\mathrm{d}\vec{\xi}\,\frac{\mathrm{d}^2 n^{(j)}}{\mathrm{d}\vec{\xi}\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\tau_{\mathrm{c}}}\frac{\mathrm{d}V_{\mathrm{c}}}{\mathrm{d}z}\int_{\ln f_{\mathrm{orb}}-\Delta\ln f_{\mathrm{orb}}/2}^{\ln f_{\mathrm{orb}}+\Delta\ln f_{\mathrm{orb}}/2}\mathrm{d}\ln f_{\mathrm{orb}}\frac{\mathrm{d}\tau_{\mathrm{c}}}{\mathrm{d}\ln f_{\mathrm{orb}}}\\
&= \frac{1}{V_{\mathrm{sim}}}\left[\frac{\mathrm{d}z}{\mathrm{d}\tau_{\mathrm{c}}}\frac{\mathrm{d}V_{\mathrm{c}}}{\mathrm{d}z}\right]_{z_j}\Delta\tau_{\mathrm{c}}^{(j)}(f_{\mathrm{orb}}).
\end{aligned}$$

$$h_{\mathrm{c}}^2(f) \simeq \sum_j\sum_k N_{\mathrm{i}}^{(j)}(f_{\mathrm{orb}}^{(k)}) \times h_{\mathrm{c},1}^2\left(f; z_j, \vec{\xi}_j, f_{\mathrm{orb}}^{(k)}\right)$$

Example: Hydrodynamical cosmological simulations